Rajshahi University journal of science and engineering **ISSN 2309-0952 (Print)**
Vol. 46-53 (2018-2025), pp 1-9 **ISSN 2408-8803 (Online)**
DOI: https://doi.org/10.64102/rujse.0748

An asymptotic method for periodic solution of nonlinear autonomous systems modeled by second order hyperbolic type partial differential equation

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Abstract

In this article, a new perturbation is developed and used to find analytical solutions which describe the time response of nonlinear systems governed by second order hyperbolic type partial differential equations, based on the pioneer work of Krylov-Bogoliubov-Mitropolskii (KBM). In addition, a numerical code is also developed to test the accuracy of analytical solution. Finally, the method is applied to longitudinal vibration of a rod in which nonlinear behavior occurs. The analytical solution shows an excellence coincidence with numerical solution.

Keywords: Asymptotic method, nonlinearity, periodic solution

1. Introduction

Prior to the advent of advanced digital computers, engineers and applied mathematicians relied on exact or numerical solutions of differential equations. In many cases, the exact solutions often required a great deal of effort and mathematical sophistication. In addition, many physical systems cannot be solved directly. That is why; the method of analytical solution which is an approximation technique is developed by using linearizations, simple geometric representations and other idealizations. Although the analytical solutions are elegant and yield insight, those solutions are limited with respect to how faithfully they represent the real systems-especially those are nonlinear and irregularly shaped.

The method of Krylov-Bogoliubov-Mitropolskii (KBM) [1, 2] is one of the widely used approximation techniques to obtain analytical solutions of second order weakly nonlinear ordinary differential equations. The method was originally developed for obtaining periodic solution of second-order ordinary differential equation with small nonlinearities. Later this method has been extended to nonlinear damped-oscillatory system by Popov [3]. Murty et al. [4] has been developed an asymptotic method to obtain the response of nonlinear over-damped system. Sattar [5] has found an asymptotic solution of a critically damped nonlinear system. Similar system for partial differential equations is neither trivial nor simple. The KBM method was extended to partial differential equation with small nonlinearities by Mitropolskii and Moseenkov [6]. Bojadziev and Lardner [7] have extended the KBM method to hyperbolic-type partial differential equation of the form:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\chi(x)\frac{\partial u}{\partial x}\right) + \varepsilon F\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \varepsilon\right) \tag{1}$$

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where ε is a small parameter and *F* is a given nonlinear function. Bojadziev and Lardner [7] mainly investigated the monofrefquent solution of Eq. (1). Islam [8] studied unified KBM method for solving hyperbolic-type partial differential equations. Alam et. al. [9] presents a general formula to investigate a class of nonlinear partial differential equations. Recently, Islam [10] investigates Eq. (1) in the presence of internal resonance of 1:2. In the present paper, the periodic solution of Eq. (1) is presented based on both analytical and numerical point of view.

2. The Method

Let us consider that $u(x,t)$ satisfies a pair of homogeneous boundary conditions involving $u(x,t)$ and its derivatives at $x=0$ and $x=l$:

$$B_j(u)=\beta_{j1}u(0,t)+\beta_{j2}\frac{\partial u(0,t)}{\partial x}+\beta_{j3}u(l,t)+\beta_{j4}\frac{\partial u(l,t)}{\partial x},\ j=1,2 \tag{2}$$

where β_{jr}, $j=1,2$ and $r=1,2,3,4$ are eight constants. First of all, let us consider the unperturbed system of Eq. (1) with the boundary conditions in Eq. (2) of the form:

$$\rho(x)\frac{\partial^2 u_0}{\partial t^2}=\frac{\partial}{\partial x}\left(\chi(x)\frac{\partial u_0}{\partial x}\right), B_j\left(u^{(0)}\right)=0,\ j=1,2 \tag{3}$$

We seek separable solution of Eq. (3) of the form $u^{(0)}(x,t)=\varphi_n(x)T(t)$. Substituting $u^{(0)}(x,t)$ into Eq. (3) one may obtain the following equations:

$$\frac{d^2T(t)}{dt^2}+\lambda_n^2 T(t)=0 \tag{4}$$

and

$$\frac{d}{dx}\left(\chi(x)\frac{d}{dx}(\varphi_n(x))\right)+\lambda_n^2\rho(x)\varphi_n(x)=0, B_j(\varphi_n(x))=0,\ j=1,2 \tag{5}$$

The Strum-Liouville system in Eq. (5) determines the set of eigen-value $\{\lambda_n\}$ and eigen function $\{\varphi_n(x)\}$ which form a complete set of functions, orthogonal with respect to the weight function $\rho(x)$ provided that the boundary conditions $B_j(\varphi_n(x))$ satisfy the self-adjointness condition. Suppose that $\varphi_n(x)$ be normalized i.e.

$$\int_0^l \rho(x)\varphi_m(x)\varphi_n(x)dx=\delta_{m,n} \tag{6}$$

where $\delta_{m,n}$ is the Kronecker delta symbol. In general, Strum-Liouville system has the property that an infinite number of frequencies occur. The general solution of the generating system can be found as a sum of these separable solutions. The complete set of separable solutions can be written in the form:

$$\varphi_n(x)a_n\sin(\omega_n t+\alpha_n),\, n=1,2,\cdots \tag{7}$$

For simplicity we assume that only first frequency occurred (i.e. the lowest positive frequency). We wish to find the solution of Eq. (1) which corresponds to the frequencies ω_1. Using the KBM method, we seek a solution in the form of an asymptotic series in ε of the form:

$$u(x,t,\varepsilon)=\varphi_1(x)a\sin\psi+\varepsilon u_1(x,a,\psi)+\mathrm{O}(\varepsilon^2) \tag{8}$$

where a and ψ are the function of t defined by the following first order differential equations

$$\begin{aligned} a^{\&} &= \varepsilon A(a)+O(\varepsilon^2) \\ \Psi^{\&} &= \omega_1+\varepsilon B(a)+O(\varepsilon^2) \end{aligned} \tag{9a,b}$$

Substituting Eq. (8) into Eq. (1), utilizing Eq. (9a, b) and comparing the coefficients of ε, one may obtain

$$\rho(x)\varphi_1(x)(2\omega_1 A\cos\psi-2a\omega_1 B\sin\psi)+\rho(x)\frac{\partial^2 u_1}{\partial\psi^2}=\frac{\partial}{\partial x}\left(\chi(x)\frac{\partial u_1}{\partial x}\right)+f^{(0)} \tag{10}$$

where $F^{(0)}=f(x,u^{(0)},u_x^{(0)},\cdots)$ and $u^{(0)}=\varphi_1(x)a\sin\psi$. Let us expand $u_1(x,a,\psi)$ as a Fourier series in x using the basis $\{\varphi_n(x)\}$ in the following form:

$$u_1(x,a,\psi)=\sum_{j=1}^{\infty}v_j(a,\psi)\varphi_j(x) \tag{11}$$

Substituting Eq. (11) into Eq. (10), multiplying both sides by $\varphi_s(x)$ and integrating with respect to x within the limits 0 to l with the aid of Eq. (5) and Eq. (6) gives

$$(2\omega_1 A\cos\psi-2a\omega_1 B\sin\psi)\delta_{1,s}+\left(\omega_1^2\frac{\partial^2}{\partial\psi^2}-\omega_s^2\right)v_s=F_s(a,\psi) \tag{12}$$

with

$$F_s(a,\psi)=\int_0^l F^{(0)}\varphi_s(x)dx \tag{13}$$

In order to determine the unknown functions A, B and $v_s, s=1,2,\cdots$ it is assumed that v_s does not include the fundamental terms, so that the coefficients of the expansion of v_s do not become large. It is also assumed that $F_s(a,\psi)$ can be expanded in a Taylor series as:

$$\begin{aligned} F_s(a,\psi)=F_{s,0}&+F_{s,1}\cos\psi+F_{s,2}\cos 2\psi+\cdots \\ &+G_{s,1}\sin\psi+G_{s,2}\sin 2\psi+\cdots \end{aligned} \tag{14}$$

Substituting Eq. (14) into Eq. (12) and assuming that v_s excludes the terms $\cos\psi$ and $\cos\psi$. Hence, one may obtain

$$\begin{aligned} &2\omega_1 A = F_{1,1} \\ &-2a\omega_1 B = G_{1,1} \\ &\left(\omega_1^2 \frac{\partial^2}{\partial\psi^2} - \omega_1^2\right) v_1 = F_{1,0} + F_{1,2}\cos 2\psi + \cdots + G_{1,2}\sin 2\psi + \cdots \end{aligned} \tag{15a-c}$$

and

$$\left(\omega_1^2 \frac{\partial^2}{\partial\psi^2} - \omega_s^2\right) v_s = F_{s,0} + F_{s,2}\cos 2\psi + \cdots + G_{s,2}\sin 2\psi + \cdots \tag{16}$$

According to the KBM method, particular solutions of Eqs. (15a-c) give the unknown functions A, B and v_1 which complete the determination of the first order solution of Eq. (1). The method can be carrying to higher order approximations in a similar way.

3. Example

Let us consider that $u(x,t)$ is the displacement of the rod at position x, $\sigma(x,t)$ is the longitudinal stress at that point and $e(x,t)$ is the longitudinal strain. Then the longitudinal vibration of a nonlinear elastic rod is described by the equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial\sigma}{\partial x} \tag{17}$$

The stress-strain relation is assumed in the form

$$\sigma = Ke + \frac{1}{3}\varepsilon E e^3 \tag{18}$$

where K is modulus, $e = \dfrac{\partial u}{\partial x}$ is axial strain and the term with E represents nonlinear elastic behavior. Eliminating σ from Eq. (17) and Eq. (18) one may get a partial differential equation in the form

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = K u_{xx} + \varepsilon E u_x^2 u_{xx} \tag{19}$$

Let us consider the boundary conditions

$$u(0,t) = 0,\; h u_x(l,t) + u(l,t) = 0 \tag{20a,b}$$

Applying the boundary conditions in Eq. (20a,b), one gets the eigen-functions and corresponding eigen values of the unperturbed Eq. (19) as follows:

$$\varphi_n(x)=c_n\sin(p_n x),\ \omega_n^2=\frac{K\lambda_n^2}{\rho},\ n=1,2,\cdots \tag{21a,b}$$

where $\{\omega_n\}$ are the root of equation

$$\tan(pl)=-hp \tag{22}$$

and the constant $\{c_n\}$ satisfy the relation

$$\frac{\rho c_n^2}{2}\left(l+\frac{h}{\left(1+h^2\omega_n^2\right)}\right)=1,\ n=1,2,\cdots \tag{23}$$

In Eq. (19) the nonlinear function is given by $F=Eu_x^2u_{xx}$. Therefore $F_1(a,\psi)$ can be written as

$$F_1(a,\psi)=E_1^{11}a^3\sin^3\psi=\frac{E_1^{11}a^3}{4}\{3\sin\psi-\sin(3\psi)\} \tag{24}$$

where $E_s^{11}=E\int_0^l(\varphi_1'(x))^2\varphi_1''(x)\varphi_s(x)dx$. Therefore only the non-zero coefficients of $F_{s,n}$ and $G_{s.n}, n=1,2\cdots$ are $G_{1,1}=\frac{3E_s^{11}a^3}{4}$ and $G_{1,3}=-\frac{3E_s^{11}a^3}{4}\sin(3\psi)$. Substituting the above values into Eqs. (15a-c) and Eq. (16) and solving them, one may get

$$A=0$$
$$B=-\frac{3E_1^{11}a^2}{8\omega_i} \tag{25a-c}$$
$$v_1=-\frac{3E_1^{11}a^3}{40\omega_1^2}\sin(3\psi)$$

and

$$v_s=0, s\geq 2 \tag{26}$$

Again, substituting the values of A and B from Eqs. (25a,b) into Eq. (9a,b) and integrating them with respect to t, one may obtain

$$a=a_0,\ \psi=\psi_0+\omega_1 t-\frac{3E_i^{11}a^2t}{8\omega_i} \tag{27a,b}$$

Thus, the first order solution of Eq. (19) is given by

$$u(x,t,\varepsilon)=\varphi_1(x)[a\sin\psi+\varepsilon v_1(a,\psi)] \tag{28}$$

where a, ψ and v_1 are given by the Eqs.(27a,b) and Eq. (25c) respectively.

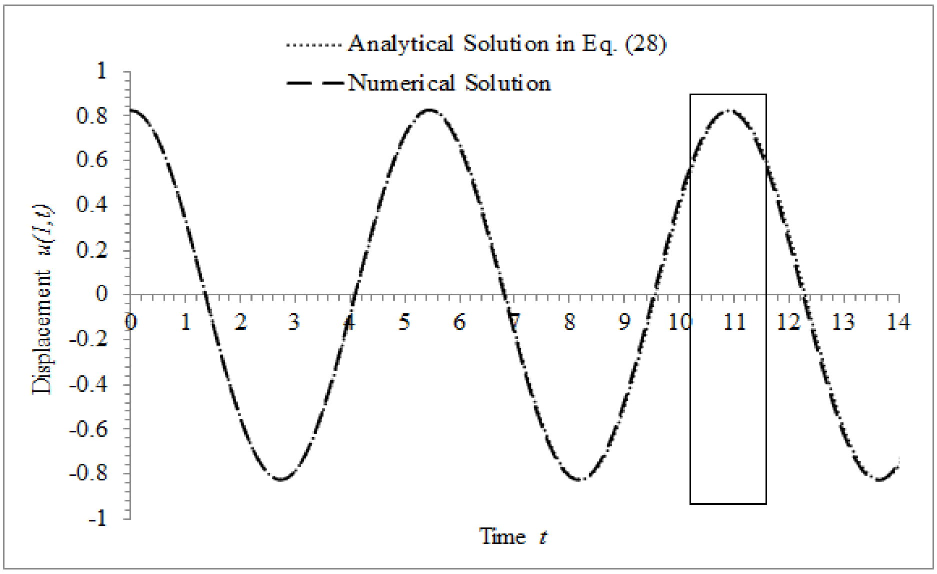

Fig.1a: Analytical solution and the corresponding numerical solution of Eq. (19) at $x=1$ are plotted. Here a and ψ are computed by Eq. (27a,b) with $\varepsilon=0.2$, $a_0=0.999062$, $\psi_0=1.570796$, $\omega_1=1.144465$, $\rho=1.0$, $K=1.0$, $l=2.0$ or initial conditions $[u(x,0)=c_1\sin(\omega_1 x), u_t(x,0)=0]$.

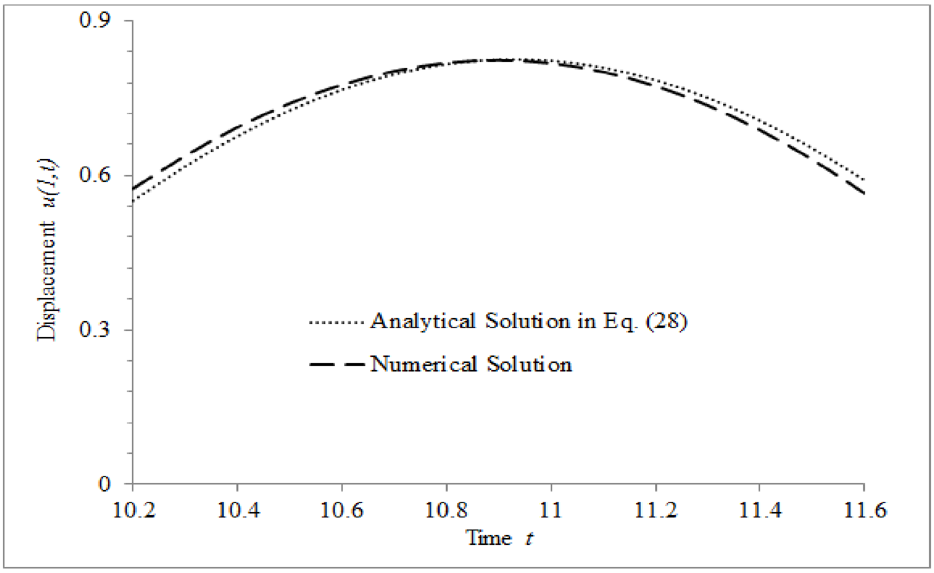

Fig.1b: Partial enlargement of Fig.1a

Table 1: Percentage error between analytical and numerical solution at $x=1$ for $10.2 \le t \le 11.6$ (i.e. for partial enlargement in Fig.1b)

t	u_{an}	u_{nu}	$error=\left(\frac{u_{nu}-u_{an}}{u_{nu}}\right)*100$
10.2	0.551021	0.574628	4.108
10.4	0.676681	0.69387	2.277
10.6	0.766792	0.776515	1.252
10.8	0.816397	0.818584	0.257
11.0	0.822735	0.817914	-0.589
11.2	0.78545	0.773878	-1.495
11.4	0.706624	0.678166	-2.68
11.6	0.59061	0.565342	-4.469

Here, u_{an} and u_{nu} are the analytical and numerical solutions respectively.

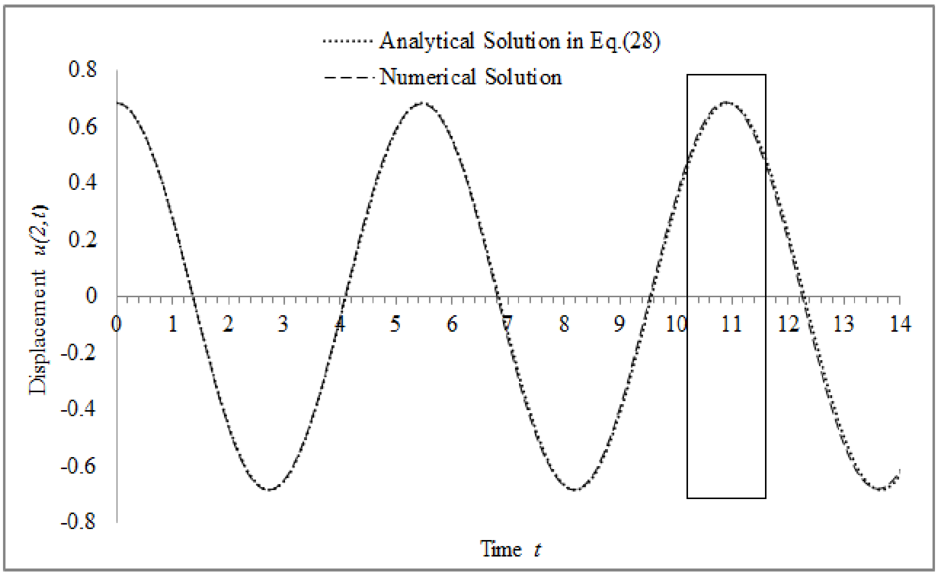

Fig.2a: Analytical solution and the corresponding numerical solution of Eq. (19) at $x=2$ are plotted. Here a and ψ are computed by Eq. (27a,b) with $\varepsilon=0.2$, $a_0=0.999062$, $\psi_0=1.570796$, $\omega_1=1.144465$, $\rho=1.0$, $K=1.0$, $l=2.0$ or initial conditions $[u(x,0)=c_1\sin(\omega_1 x), u_t(x,0)=0]$.

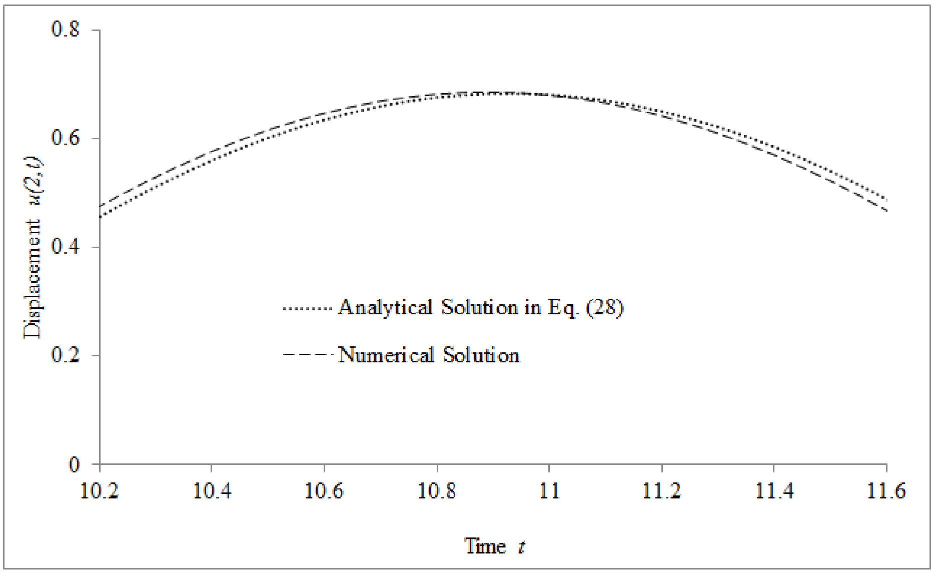

Fig.2b: Partial enlargement

Table 2: Percentage error between analytical and numerical solution at $x = 2.0$ for $10.2 \le t \le 11.6$ (i.e. for partial enlargement in Fig.2b)

t	u_{an}	u_{nu}	$error = \left(\frac{u_{nu} - u_{an}}{u_{nu}}\right) * 100$
10.2	0.45573	0.475126	4.082
10.4	0.55966	0.576049	2.845
10.6	0.634187	0.646216	1.861
10.8	0.675214	0.68142	0.910
11.0	0.680456	0.679684	-0.113
11.2	0.649619	0.641438	-1.275
11.4	0.584425	0.569331	-2.651
11.6	0.488472	0.467691	-4.44

Here, u_{an} and u_{nu} are the analytical and numerical solutions respectively.

4. Results

An asymptotic method is developed to obtain the analytical solution of autonomous nonlinear differential system modeled by hyperbolic type partial differential equation. As very limited research on second order hyperbolic partial differential equation is found, we have developed numerical code to solve the original governing equation of motion to test the accuracy of analytical solution. The analytical solution in Eq. (28) is useful for periodic solution of Eq. (19). We are interested to compare the analytical solution with numerical solution generated by finite difference method. For numerical solution of Eq. (19), we set $\rho = 1, K = 1, l = 2$ and $h = 1$ so the first solution of Eq. (22) is 1.144465 and hence $\lambda_1 = 1.14465$. For $\varepsilon = 0.2$ with initial values $[u(x,0) = c_1 \sin(\omega_1 x), u_t(x,0) = 0]$, $u(x,t,\varepsilon)$ has been evaluated. The result at $x = 1$ (i.e. at the middle point of rod) is presented in Fig. 1a. A partial enlargement for $10.2 \le t \le 11.6$ is also shown in Fig. 1b and the corresponding percentage error is presented in Table 1. Similar results at $x = 2$ (i.e. at the lower end of rod) are presented in Fig. 2a, Fig. 2b and Table 2 respectively. Both Tables insure that the absolute percentage error is less than 5.0%. From the Figures, it is clear that the analytical solution in Eq. (28) shows a good coincidence with numerical solution.

Acknowledgement

The author is grateful to the potential reviewers for their helpful comments/suggestions to prepare the revised manuscript.

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